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EXACT PROBABILITY DENSITY FUNCTION OF THE MONOPULSE RATIO

U.S. ARMY MISSILE RESEARCH AND DEVELOPMENT COMMAND

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It has been shown that unless there is only a single on-axis target, the angle estimator (monopulse ratio) at the output of a monopulse radar is biased. No matter how large the signal-to-noise ratio, the variance of the estimate is unbounded. A measure of the spread of the probability density function has been introduced and a procedure developed to calculate the spread from the exact closed form expression for the probability density function of the monopulse ratio. The bias is also available by numerically calculating the mean of the probability density function.

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CONTENTS

I. INTRODUCTION

The signals at the input to the difference and sum channels of a monopulse receiver are (in complex envelope notation)

$$D(t) = \left[g_D\ (\theta)\ g_S\ (\theta)\ A + N_D\right] e^{i\omega t} \tag{1}$$

$$S(t) = \left[g_S\ (\theta)\ g_S(\theta)\ A + N_S\right] e^{i\omega t} \tag{2}$$

where $g_D(\theta)$ and $g_S(\theta)$ are the one way voltage gains (for off-boresight angle, θ) of the difference and sum antennas; A is the voltage return at an isotropic antenna (it includes transmitter power, target range, target backscatter coefficient, etc.); and $N_D(t)$ and $N_S(t)$ are zero-mean Gaussian receiver noises. It is assumed in Equations (1) and (2) that the target has been illuminated by the sum pattern and received by both difference and sum patterns.

Alternatively, the following can be written:

$$D_R = \text{Real } D(t) = x \cos \omega t - y \sin \omega t \tag{3}$$

$$S_R = \text{Real } S(t) = u \cos \omega t - v \sin \omega t \tag{4}$$

where x, y and u, v are the in-phase, quardrature components in the difference and sum channels, respectively.

The monopulse ratio is defined as

$$r_1 \triangleq \text{Real } \left\{\frac{D(t)}{S(t)}\right\} = \frac{xu + yv}{u^2 + v^2} = \frac{\int_{LPF} D_R S_R dt}{\int_{LPF} S_R^2 dt} \quad . \tag{5}$$

Clearly, when there is a single target and no noise,

$$r_1 = \frac{g_D(\theta)}{g_S(\theta)} \quad . \tag{6}$$

Since the right side is a monotonic function of θ (to avoid ambiguities) in the region of interest, measuring r_1 yields θ, the angle of the target with respect to boresight. This result is independent of target amplitude fluctuations which are manifest in A.

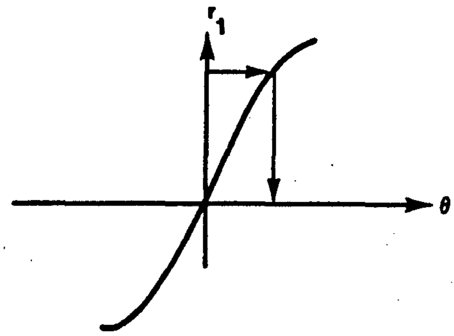

Figure 1.

When noise is present, the following is obtained:

$$r_1 = \frac{(\bar{x} + n_x)\ (\bar{u} + n_u) + (\bar{y} + n_y)\ (\bar{v} + n_v)}{(\bar{u} + n_u)^2 + (\bar{v} + n_v)^2} \tag{7}$$

where the over bar represents a statistical mean caused by the presence of the target. Because the phase reference is arbitrary, the target phase can be used as a reference i.e., the target is set completely in the in-phase components; then,

$$r_1 = \frac{(\bar{x} + n_x)\ (\bar{u} + n_u) + n_y\ n_v}{(\bar{u} + n_u)^2 + n_v^2} \tag{8}$$

II. APPROXIMATE ANALYSIS

At this point an approximate analysis is usually made which goes as follows: Since the signal-to-noise ratio is large, the quadrature terms (which have no signal) compared to the in-phase terms can be neglected:

$$r_1 \approx \frac{\bar{x} + n_x}{\bar{u} + n_u} \tag{9}$$

so that r_1 is the ratio of two Gausssians. Again, using the fact that the signal-to-noise ratio is large, i.e.,

$$\bar{u}^2 >> \overline{n_u^2} \tag{10}$$

the noise in the denominator of Equation (9) compared to the signal there can be neglected; thus,

$$r_1 \approx \frac{\bar{x}}{\bar{u}} + \frac{n_x}{\bar{u}} \quad . \tag{11}$$

and because n_x is zero mean

$$\bar{r}_1 = \frac{\bar{x}}{\bar{u}} \tag{12}$$

i.e., the estimate is unbiased, and

$$\sigma_{r_1}^2 = \frac{\overline{n_x^2}}{\bar{u}^2} \quad . \tag{13}$$

Since $\bar{u}^2$ is the total sum signal power and $\overline{n_x^2}$ is half the noise difference power, [Equation (8)], the accuracy of the monopulse measurement is taken as

$$\sigma_{r_1} = \frac{1}{\sqrt{2\ S/N}} \quad . \tag{14}$$

This approximate analysis linearizes the monopulse ratio and states that the average of a quotient equals the quotient of the averages (of numerator and denominator). Clearly, Equation (12) cannot be valid because by decreasing n_u in Equation (9) by a small amount, ϵ, r_1 is increased more than it is decreased by increasing n_u by an equivalent amount. Thus, a bias must exist. A further critique of the passage from Equation (9) to Equation (11) is in order. The validity of Equation (10) does not allow the assumption that n_u, the instantaneous value of the noise, is always less than $\bar{u}$. After all, the noise is Gaussian and can assume all values.

In the exact analysis given in the following equations, the bias will be made manifest and no assumptions concerning large signal-to-noise ratio will be invoked.

III. EXACT ANALYSIS

Equations (1) and (2) can be rewritten for the situation of multiple targets as

$$D(t) = \left(\sum_{n=1}^{N} g_D(\theta_n)\, g_S(\theta_n)\, A_n + N_D \right) e^{i\omega t} \tag{15}$$

$$S(t) = \left(\sum_{n=1}^{N} g_S(\theta_n)\, g_S(\theta_n)\, A_n + N_S \right) e^{i\omega t} \tag{16}$$

The noise-free monopulse ratio is defined:

$$\hat{r} \triangleq \mathrm{Re} \left[\frac{\sum_{n=1}^{N} r_n B_n}{\sum_{n=1}^{N} B_n} \right]$$

where

$$B_n \triangleq g_S^2 (\theta_n)\, A_n$$

is the sum channel return from the n^{th} target and

$$r_n \triangleq \frac{g_D(\theta_n)}{g_S(\theta_n)}$$

is the monopulse ratio associated with the n^{th} target. The random phases of the targets are included in the complex numbers, A_n, and give rise to glint, but in this analysis they are treated as constants.

As distinct from the single target case, the following no longer applies:

$$\frac{\overline{x}}{\overline{u}} = \frac{\overline{y}}{\overline{v}} \quad .$$

The following auxilliary variable is introduced:

$$\zeta_{x,u,y,v}(r) \triangleq xu + yv - r(u^2 + v^2) \quad ; \tag{21}$$

then,

$$\Pr.\ (r_1 \le r) \equiv \Pr.\ (\zeta \le 0) \quad , \tag{22}$$

thus the probability density function (PDF) of r_1 is given in terms of the PDF of ζ by

$$p_{r_1}(r) = \frac{d}{dr} \int_{-\infty}^{0} p_{\zeta}(\alpha)\, d\alpha \quad . \tag{23}$$

Note that ζ does not involve a ratio.

The Fourier transform of $p_{\zeta}(\alpha)$ is defined by

$$F_{\zeta}(\omega) \stackrel{\Delta}{=} \int_{-\infty}^{\infty} p_{\zeta}(\alpha)\ e^{-i\alpha\omega} d\alpha \tag{24}$$

and the Fourier integral theorem states

$$p_{\zeta}(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F_{\zeta}(\omega) e^{+i\alpha\omega} d\alpha \quad . \tag{25}$$

It appears at first glance that to find p_{ζ}, F_{ζ} is needed and conversely to find F_{ζ}, p_{ζ} is needed — so what is the use of introducing the Fourier transform? However, Equation (24) states that $F_{\zeta}(\omega)$ is the expectation of $e^{-i\zeta\omega}$. This may be calculated over the domain of the random variables x, u, y, v. Thus,

$$F_{\zeta}(\omega) = \iiiint_{-\infty}^{\infty} e^{-i\omega[xu + yv - r(u^2 + v^2)]}\ p_{x,u,y,v}(x,u,y,v)\, dx \ldots dv. \tag{26}$$

The in-phase and quadrature noise components in each channel are independent and identically distributed. Independence between the noises in the difference and sum channels of the receiver is assumed so that the joint density in Equation (26) becomes the product of the four densities

$$p_x(x) = \frac{1}{\sqrt{2\pi}\, a}\ e^{\frac{-(x-\bar{x})^2}{2a^2}} \tag{27}$$

$$p_y(y) = \frac{1}{\sqrt{2\pi}\, a} e^{\frac{-(y-\bar{y})^2}{2a^2}} \tag{28}$$

$$p_u(u) = \frac{1}{\sqrt{2\pi}\, b} e^{\frac{-(u-\bar{u})^2}{2b^2}} \tag{29}$$

$$p_v(v) = \frac{1}{\sqrt{2\pi}\, b} e^{\frac{-(v-\bar{v})^2}{2b^2}} \tag{30}$$

thus

$$p_{x,u,y,v}(x,u,y,v) = p_x(x)\; p_u(u)\; p_y(y)\; p_v(v) \quad . \tag{31}$$

Hence

$$F_\zeta(\omega) = \iint_{-\infty}^{\infty} e^{-i\omega(xu-rv^2)}\; p_x(x)\; p_u(u)\; dx\; du \quad .$$

$$\cdot \iint_{-\infty}^{\infty} e^{-i\omega(yv-rv^2)}\; p_y(y)\; p_v(v)\; dy\; dv \quad . \tag{32}$$

The double integral in x, u is evaluated by completing the squares in the exponent and yields

$$\iint_{-\infty}^{\infty} (\;)dx\, du = \frac{e^{-\frac{\bar{u}^2}{2b^2}\left[\left(\frac{b}{a}\,\frac{\bar{x}}{\bar{u}}\right)^2+1\right]}\; e^{-\frac{\bar{u}^2}{2b^2}\cdot\frac{2iab\omega\left[\left(\frac{b}{a}\,\frac{\bar{x}}{\bar{u}}\right)^2\frac{br}{a}+\frac{b}{a}\,\frac{\bar{x}}{\bar{u}}\right]-\left[\frac{b}{a}\,\frac{\bar{x}}{\bar{u}}\right]^2+1}{\sqrt{a^2b^2\omega^2-2i\frac{b}{a}rab\omega+1}}}}{\sqrt{a^2b^2\omega^2-2i\frac{b}{a}rab\omega+1}} \tag{33}$$

The double integral in y, v is obtained from Equation (33) by replacing $\bar{x}$, $\bar{u}$ by $\bar{y}$, $\bar{v}$, respectively. Thus

$$F_\xi(\omega) = \frac{e^{-\frac{1}{2b^2}\left[\bar{u}^2\hat{s}_1^2+\bar{v}^2\hat{s}_2^2+\bar{u}^2+\bar{v}^2\right]}}{a^2b^2\omega^2-2isab\omega+1} \quad .$$

$$\cdot\; e^{-\frac{2iab\omega\left[\left(\bar{u}^2\hat{s}_1^2+\bar{v}^2\hat{s}_2^2\right)s+\bar{u}^2\hat{s}_1+\bar{v}^2\hat{s}_2\right]-\left[\bar{u}^2\hat{s}_1^2+\bar{v}^2\hat{s}_2^2+\bar{u}^2+\bar{v}^2\right]}{2b^2\left[a^2b^2\omega^2-2isab\omega+1\right]}} \tag{34}$$

where

$$s(r) \triangleq \frac{br}{a} \tag{35}$$

$$\hat{s}_1 \triangleq \frac{b}{a}\frac{\bar{x}}{\bar{u}} \tag{36}$$

$$\hat{s}_2 \triangleq \frac{b}{a}\frac{\bar{y}}{\bar{v}} \quad . \tag{37}$$

When only a single target is present, $\hat{s}_1 = \hat{s}_2$.

In the absence of noise the monopulse ratio is given by

$$\hat{r} \triangleq \frac{\bar{x}\,\bar{u} + \bar{y}\,\bar{v}}{\bar{u}^2 + \bar{v}^2} \tag{38}$$

Define also

$$\hat{s} \triangleq \frac{b\hat{r}}{a} = \frac{\bar{u}^2\hat{s}_1 + \bar{v}^2\hat{s}_2}{\bar{u}^2 + \bar{v}^2} \tag{39}$$

and

$$\hat{t}^2 \triangleq \frac{\bar{u}^2\hat{s}_1^2 + \bar{v}^2\hat{s}_2^2}{\bar{u}^2 + \bar{v}^2} = \hat{s}^2 + \frac{\bar{u}^2\bar{v}^2(\hat{s}_1 - \hat{s}_2)^2}{(\bar{u}^2 + \bar{v}^2)^2} \geq 0 \quad . \tag{40}$$

For a single target $\hat{t} = \hat{s}$.

With the aid of Equations (38), (39), and (40), Equation (34) simplifies to

$$F_\zeta(\omega) = \frac{e^{-x(\hat{t}^2 + 1)}\, e^{-x\frac{2iab\omega(\hat{t}^2 s + \hat{s}) - (\hat{t}^2 + 1)}{a^2b^2\omega^2 - 2isab\omega + 1}}}{a^2b^2\omega^2 - 2isab\omega + 1} \tag{41}$$

where the sum channel signal-to-noise ratio has been introduced:

$$x \triangleq \frac{\bar{u}^2 + \bar{v}^2}{2b^2} \quad . \tag{42}$$

Since this is not a function of $s - \hat{s}$ unless $\hat{s} = 0$, or from Equation (39) $\overline{x}/\overline{u} = 0 = \overline{y}/\overline{v}$, it was concluded that unless there is a single target on boresight, there will be a bias in the estimate of r_1. Substituting Equation (25) into Equation (23), the following is obtained:

$$p_{r_1}(r) = \frac{b}{a} \int_{-\infty}^{0} \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{d}{ds} F_{\zeta}(\omega)\, e^{i\omega\zeta}\, d\omega \right] d\zeta \tag{43}$$

or

$$p_{r_1}(r) = \frac{b}{a} \int_{-\infty}^{0} \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{d}{ds} F_{\zeta} \left(\frac{\omega}{ab}\right) e^{\frac{i\omega}{ab}\zeta} \frac{d\omega}{ab} \right] d\zeta \quad . \tag{44}$$

Since $F_{\zeta}(\omega/ab)$ is given by Equation (41), the differentiation performed and the following is obtained:

$$p_{r_1}(r) = \frac{2b}{a}\, e^{-x(\hat{t}^2+1)} \int_{-\infty}^{0} \frac{1}{2\pi} \int_{-\infty}^{\infty} i\omega \left[1 - x\hat{t}^2 - x\frac{2i\omega(\hat{t}^2 s + \hat{s}) - (\hat{t}^2+1)}{\omega^2 - 2is\omega + 1} \right]$$

$$\cdot \frac{e^{-x\frac{2i\omega(\hat{t}^2 s + \hat{s}) - (\hat{t}^2+1)}{\omega^2 - 2is\omega + 1}}}{(\omega^2 - 2is\omega + 1)^2}\, e^{\left(\frac{i\omega}{ab}\right)\zeta} \left(\frac{d\omega}{ab}\right) d\zeta \tag{45}$$

In carrying out the ω integration, the factor $i\omega/ab$ corresponds to differentiation with respect to ζ; this combines with the subsequent integration over ζ from $-\infty$ to 0 so that the result of the ω integration needs to be evaluated only at $\zeta = 0$. Thus in terms of

$$G(\tau) \triangleq \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-x\tau\left(\frac{2i\omega(\hat{t}^2 s + \hat{s}) - (\hat{t}^2+1)}{\omega^2 - 2is\omega + 1}\right)}}{(\omega^2 - 2is\omega + 1)^2}\, d\omega \tag{46}$$

the PDF of the monopulse ratio becomes

$$p_{r_1}(r) = \frac{2b}{a}\, e^{-x(\hat{t}^2+1)} \left[(1 - x\hat{t}^2) G(\tau) + G'(\tau) \right]_{\tau=1} \quad . \tag{47}$$

In order to evaluate $G(\tau)$, the integrand is written as a product in terms of the essential singularities at

$$\omega_{1,2} = i(s \pm \sqrt{s^2 + 1}) \quad . \tag{48}$$

Then,

$$G(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-\frac{i\alpha\tau}{\omega-\omega_1}}}{(\omega-\omega_1)^2} \cdot \frac{e^{\frac{i\beta\tau}{\omega-\omega_2}}}{(\omega-\omega_2)^2}\, d\omega \tag{49}$$

where

$$\alpha \triangleq \frac{\chi}{2\sqrt{s^2+1}} \left\{ \left[\hat{s}\left(s + \sqrt{s^2+1}\right) + 1\right]^2 + (\hat{t}^2 - \hat{s}^2)\left(s + \sqrt{s^2+1}\right)^2 \right\} \geq 0 \tag{50}$$

$$\beta \triangleq \frac{\chi}{2\sqrt{s^2+1}} \left\{ \left[\hat{s}\left(s - \sqrt{s^2+1}\right) + 1\right]^2 + (\hat{t}^2 - \hat{s}^2)\left(s - \sqrt{s^2+1}\right)^2 \right\} \geq 0 \; . \tag{51}$$

Since ω_1 lies in the upper half plane and ω_2 lies in the lower half plane, it is easily shown by completing the contour and employing a Laurent expansion that

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{\frac{-i\alpha\tau}{\omega - \omega_1}}}{(\omega - \omega_1)^2}\, e^{i\omega t}\, d\omega = \frac{1 + \text{sgn}\,t}{2}\, (-1)\, e^{i\omega_1 t}\left(\frac{t}{\alpha\tau}\right)^{1/2} I_1\left(2\sqrt{\alpha\tau t}\right) \tag{52}$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{\frac{i\beta\tau}{\omega - \omega_2}}}{(\omega - \omega_2)^2}\, e^{i\omega t}\, d\omega = \frac{1 - \text{sgn}\,t}{2}\, (-1)\, e^{i\omega_2 t}\left(\frac{-t}{\alpha\tau}\right)^{1/2} I_1\left(2\sqrt{-\beta\tau t}\right) \tag{53}$$

where I_1 denotes the modified Bessel function.

Thus using Parseval's theorem, Equation (49) becomes

$$G(\tau) = \int_0^\infty e^{-2\sqrt{s^2+1}\,t} \left(\frac{t}{\alpha\tau}\right)^{1/2} \left(\frac{t}{\beta\tau}\right)^{1/2} I_1\left(2\sqrt{\alpha\tau t}\right) I_1\left(2\sqrt{\beta\tau t}\right) dt \quad . \tag{54}$$

Expanding the Bessel functions into power series and employing the Cauchy product, the following is obtained:

$$G(\tau) = \int_0^\infty e^{-2\sqrt{s^2+1}\,t} \sum_{n=0}^\infty \left[\sum_{m=0}^n \frac{(\alpha\tau)^{n-m}}{(n-m)!\,(1+n-m)!} \cdot \frac{(\beta\tau)^m}{m!\,(1+m)!}\right] t^{2+n}\, dt \quad . \tag{55}$$

Termwise, integration yields

$$G(\tau) = \frac{1}{(2\sqrt{s^2+1})^3} \sum_{n=0}^\infty (2+n)! \cdot \sum_{m=0}^n \frac{(\beta/\alpha)^m}{m!\,(1+m)!\,(n-m)!\,(1+n-m)!} \left(\frac{\alpha\tau}{2\sqrt{s^2+1}}\right)^n \quad . \tag{56}$$

Since β/α is independent of x, this yields $G(\tau)$ and hence $G'(\tau)$ and hence the PDF of the monopulse ratio as a power series in α which is proportional to the signal-to-noise ratio. This is most inconvenient for large x. To obtain a representation in closed form, it is noted that the polynomial in parentheses is a degenerate hypergeometric function[1], namely:

$$\sum_{m=0}^n \frac{(\beta/\alpha)^m}{m!\,(1+m)!\,(n-m)!\,(1+n-m)!} = \frac{1}{n!\,(1+n)!}\ {}_2F_1(-n, -n-1; 2; \beta/\alpha) \quad . \tag{57}$$

[1] Abramowitz, M. and Stegun, I. A., Handbook of Mathematical Functions, National Bureau of Standards Appl. Math. Series Vol. 55, March 1965 (Formula 15.1.1).

This is expressible in terms of a Legendre function of the first kind; the following quadratic transformation[2] is employed:

$$_2F_1(-n, -n -1; 2 ; z) = z^{-1/2} (1 - z)^{1+n} P^{-1}_{1+n} \left(\frac{1 + z}{1 - z}\right) \tag{58}$$

where P^{-1}_{1+n} is the Legendre function of the first kind of order -1 and degree 1 + n. This is valid whenever $|\arg(1 - z)| < \pi$ and z does not lie between 0 and $-\infty$. Since β/α is positive, Equation (58) holds whenever z $(= \beta/\alpha) < 1$. Assuming that $\alpha > \beta$, then Equation (58) holds. Using the relation between the Legendre functions of the first kind wit negative order and the Legendre functions of the first and second kind with positive order,[3] the following is obtained:

$$P^{-1}_{1+n} \left(\frac{1 + z}{1 - z}\right) = \frac{n!}{(2 + n)!} \; P^{1}_{1+n} \left(\frac{1 + z}{1 - z}\right) \quad ; \tag{59}$$

thus

$$\sum_{m=0}^{n} \frac{(\beta/\alpha)^m}{m!\,(1 + m)!\,(n - m)!\,(1 + n - m)!}$$

$$= \frac{1}{(2 + n)!} \; \frac{1}{(1 + n)!} \; P^{1}_{1+n} \left(\frac{\alpha + \beta}{\alpha - \beta}\right) \quad . \tag{60}$$

Since for $w > 1$,[4] the following is obtained:

$$P^{1}_{1+n} (w) = (w^2 - 1)^{1/2} \, P^{(1)}_{1+n} (w) \tag{61}$$

where $P^{(1)}_{1+n} (w)$ is the first derivative of the Legendre polynominal of degree 1 + n, $G(\tau)$ becomes

$$G(\tau) = \sum_{n=0}^{\infty} \frac{2}{(1 + n)!} \; P^{(1)}_{1+n} \left(\frac{\alpha + \beta}{\alpha - \beta}\right) \left(\frac{(\alpha - \beta)\tau}{2\sqrt{S^2 + 1}}\right)^n \quad . \tag{62}$$

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[2] Ibid. (Formula 15.4.14).

[3] Ibid. (Formula 8.2.5).

[4] Ibid. (Formula 8.6.6).

Since $G(\tau)$ is invariant with respect to an interchange of α and β [Equation (54)], if $\beta/\alpha \not> 1$, then α and β can be interchanged. Thus α-β in Equation (62) may be replaced by $|\alpha-\beta|$ and Equation (62) will hold for all α and β. Returning to Equation (47), the PDF of r_1 is written as

$$p_{r_1}(r) = \frac{b}{2a} \frac{e^{-\chi(\hat{t}^2+1)}}{\left(\sqrt{s^2+1}\right)^3} \sum_{m=1}^{\infty} \frac{(m-\chi\hat{t}^2)}{m!} P_{m(z)}^{(1)} t^{m-1} \tag{63}$$

where

$$z \triangleq \frac{\alpha+\beta}{|\alpha-\beta|} \tag{64}$$

$$t \triangleq \frac{|\alpha-\beta|}{2\sqrt{s^2+1}} \tag{65}$$

Since $P_0(z)$ is a polynominal of order zero in z, the lower summation limit in Equation (63) is extended to zero.

A generating function for the Legendre polynomials is given by[5]

$$\sum_{m=0}^{\infty} P_m(z)\frac{t^m}{m!} = e^{tz} I_0\left(t\sqrt{z^2-1}\right) \tag{66}$$

hence Equation (63) becomes

$$p_{r_1}(r) = \frac{b}{2a} \frac{e^{-\chi(\hat{t}^2+1)}}{(s^2+1)^{3/2}} \frac{1}{t} \frac{d}{dz}\left\{t\frac{d}{dt} - \chi\hat{t}^2\right\}\left[e^{tz} I_0\left(t\sqrt{z^2-1}\right)\right]. \tag{67}$$

Carrying out the t differentiation, the following is obtained:

$$p_{r_t}(r) = \frac{b}{2a} \frac{e^{-\chi(\hat{t}^2+1)}}{(s^2+1)^{3/2}} \frac{1}{t} \frac{d}{dz}\left\{e^{tz}(tz-\chi\hat{t}^2) I_0\left(t\sqrt{z^2-1}\right) + e^{tz}\left(t\sqrt{z^2-1}\right) I_1\left(t\sqrt{z^2-1}\right)\right\} \tag{68}$$

[5] Ibid. (Formula 22.9.13).

where the following relation[6] has been used:

$$\frac{d}{dt} I_0 (t) = I_1 (t) \tag{69}$$

Now let

$$w \triangleq tz = \frac{\alpha + \beta}{2\sqrt{s^2 + 1}} \tag{70}$$

so that

$$p_{r_1}(r) = \frac{b}{2a} \frac{e^{-x(\hat{t}^2+1)}}{(s^2 + 1)^{3/2}} \frac{d}{dw}\left\{e^{w}(w-x\hat{t}^2) I_0\left(\sqrt{w^2 - t^2}\right) + \right.$$
$$\left. + e^{w}\sqrt{w^2 - t^2}\; I_1\left(\sqrt{w^2 - t^2}\right)\right\} \quad . \tag{71}$$

The following function is introduced:

$$H(m,k,w) \triangleq e^{w} w^{m} \left(\sqrt{w^2 - t^2}\right)^{k} I_k \left(\sqrt{w^2 - t^2}\right) \tag{72}$$

then

$$p_{r_1}(r) = \frac{b}{2a} \frac{e^{-x(\hat{t}^2+1)}}{(s^2 + 1)^{3/2}} \frac{d}{dw} \left\{H(1,0,w) - x\hat{t}^2 H(0,0,w) + H(0,1,w)\right\} \tag{73}$$

Since[7]

$$\frac{1}{x} \frac{d}{dx} \left[x^{k} I_k(x)\right] = x^{k-1} I_{k-1}(x) \quad , \tag{74}$$

differentiation of the composite function $H(m,k,w)$ yields

$$\frac{d}{dw} H(m,k,w) = H(m,k,w) + mH(m - 1,k,w) + H(m + 1, k - 1,w) \quad . \tag{75}$$

[6] Ibid. (Formula 9.6.27).

[7] Ibid. (Formula 9.6.28).

Employing[8]

$$I_k(x) = I_{-k}(x) \tag{76}$$

and collecting terms, the following is finally obtained:

$$p_{r_1}(r) = \frac{b}{2a} \frac{e^{-x(\hat{t}^2+1)+w}}{(s^2+1)^{3/2}} \left[(2w + 1 - x\hat{t}^2)\, I_0\left(\sqrt{w^2 - t^2}\right) + \right.$$

$$\left. + \frac{(2w^2 - wxt^2 - t^2)}{\sqrt{w^2 - t^2}}\, I_1\left(\sqrt{w^2 - t^2}\right) \right] . \tag{77}$$

This is closed form solution valid for any monopulse radar, and it is suitable for computation. It yields the PDF of the monopulse ratio (which as pointed out in the text is biased), as a function of the monopulse ratio, r. If $p_{\theta_1}(\theta)$ is desired, the monopulse ratio as a function of the off-boresight angle θ, one merely starts with the one-way voltage antenna difference and sum patterns and calculates

$$r(\theta) \stackrel{\Delta}{=} \frac{g_D(\theta)}{g_S(\theta)} \tag{78}$$

(Figure 1) and its derivative, then

$$p_{\theta_1}(\theta) = p_r[r(\theta)] \left| \frac{dr}{d\theta}(\theta) \right| . \tag{79}$$

There are many papers in the literature which attempt to analyze the performance of a monopulse radar. All but one confine their attention to a linear $r(\theta)$ characteristic and assume large signal-to-noise ratio.. None come to grips with the nonlinearity inherent in the monopulse ratio and hence do not arrive at a bias or a PDF. They are limited to approximating the variance of the PDF. In order for the sceond moment to exist, i.e.,

$$\int_{-\infty}^{\infty} r^2 p_{r_1}(r)\, dr < \infty , \tag{80}$$

8. Ibid. (Formula 9.6.6).

the PDF must fall off as $|r| \to \infty$, faster than r^{-3}. For large $|s|$, α and β approach a constant times s, and w and t approach constants. Thus $p_{r_1}(r)$ falls off only as fast as r^{-3}. It is concluded that the second moment does not exist and hence that the variance is infinite for any signal-to-noise ratio. However, the distribution has a spread which may be calculated by comparison with that of a Gaussian PDF as

$$\sigma_r = \frac{R_1 - R_2}{2} \tag{81}$$

where R_1 and R_2 are given by the solution of the equations

$$\int_{-\infty}^{R_1} p_{r_1}(r)dr = 0.8413 \left(= \int_{-\infty}^{\bar{r}+\sigma G} \frac{1}{\sqrt{2\pi\sigma G}} e^{\frac{-(r-\bar{r})^2}{2\sigma G^2}} dr \right) \tag{82}$$

$$\int_{-\infty}^{R_2} p_{r_1}(r)dr = 0.1587 \left(= \int_{-\infty}^{\bar{r}-\sigma G} \frac{1}{\sqrt{2\pi\sigma G}} e^{\frac{-(r-\bar{r})^2}{2\sigma G^2}} dr \right) . \tag{83}$$

Since $p_{r_1}(r)$ given by Equation (77) is convenient to program, R_1 and R_2 are readily calculable.

The only analysis which does not ignore the fundmental nonlinearity in the monopulse ratio is one in the Soviet literature[9]. The authors employ a brute force transformation of variables, and, after integrating over three of the four variables involved, succeed in obtaining the following expression for $p_{r_1}(r)$:

$$p_{r_1}\frac{(r+\rho a)}{b} = \frac{b}{2a} \cdot \frac{e^{-\chi\left[1 + 1/2\left(\frac{bz}{a}\right)^2\right]}}{\left[1 + \left(\frac{br}{a}\right)^2\right]^{3/2}} \sum_{k=0}^{\infty} (-1)^k (2 - \delta_{k0}) \frac{(2k+1)!!}{(2k)!}$$

[9] Alexandrov, V. G. and Fedosv, V. P., "Statistical Characteristics of Signals at the Output of a Monopulse Radar Receiver with Non Fluctuating Input Signals," Academy of Sciences, U.S.S.R., Radioelectronics, Vol. 17, No. 4, April 1974 (in Russian).

$$\cdot I_k\left[\frac{x}{2}\left(\frac{bz}{a}\right)^2\right]\cdot\left\{\frac{xb^2}{2a^2}\frac{\left(\frac{a^2}{b^2}+rz\right)^2}{\frac{a^2}{b^2}+r^2}\right\}^k\cdot$$

$$\cdot\ {}_1F_1\left[k+\frac{3}{2},\ 2k+1,\ \frac{xb^2}{a^2}\ \frac{\left(\frac{a^2}{b^2}+rz\right)^2}{\frac{a^2}{b^2}+r^2}\right] \tag{84}$$

where ρ is the correlation between the difference and sum channel noises

$$z = \left[\frac{\bar{x}^2+\bar{y}^2}{\bar{u}^2+\bar{v}^2}\right]^{1/2} - \rho\frac{a}{b} \tag{85}$$

and δ_{k0} is a Kroneker delta, I_k is the modified Bessel function of order k and ${}_1F_1$ is the confluent hypergeometric function. Since each term in the infinite series involves a different Bessel function and hypergeometric function, their expression is, to say the least, unwieldy.

The procedure to be followed to calculate the PDF of r_1 is to choose the target location or locations arbitrarily and assign values to $\bar{x}$, $\bar{y}$, u, $\bar{v}$, a, b. Then Equation (42) gives x. For a selected value of r, s(r) is calculated from Equation (35), $\hat{s}_1$ and $\hat{s}_2$ from Equations (36) and (37). Then $\hat{s}$ and $\hat{t}^2$ are available from Equations (39) and (40). Equations (50) and (51) give α and β while Equations (65) and (70) give t and w. These values are employed in Equation (77) and as r is varied, the PDF of r_1 is computed.

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